

# ON DEGENERATE PLANAR HOPF BIFURCATIONS

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## **Abstract**

Our concern is the study of degenerate Hopf bifurcation of smooth planar dynamical systems near isolated singular points. To do so, we propose to split up the definition of degeneracy into two types. Degeneracy of first kind shall mean that no limit cycle surrounding the steady state can emerge after or before the critical point, with the possible emergence of limit cycles surrounding the point at infin-

ity. Degeneracy of second kind shall means that either several limit cycles or semistable cycles as a limiting case, emerge surrounding the steady state super or subcritically. In degenerate bifurcation of second kind we also show that the radius of the emerging cycle tends to zero with an "anomalous" order as the bifurcation parameter tends to the critical value. Finally, we give a sufficient condition for degenerate bifurcations of second kind up to 6-jet-equivalence, and show some "typical" forms for degenerate bifurcations.

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## 1 Introduction

The goal of this paper is the study of degenerate Hopf bifurcation (**HB**) near the critical value of a one-parameter family of smooth planar vector fields in a neighborhood of an isolated singular point. The HB is usually associated to the emergence of limit cycles after or before the critical value of the bifurcation parameter. An isolated periodic orbit of a vector field on the plane, is a limit cycle. The HB phenomenon is studied recurrently in literature, and its main details and implications for dynamical systems of higher dimensions

via the Center Manifold Theorem are gathered in the monograph [1]. Several efforts to compute generalized HB in finite dimensional systems can be found in literature (see [2] and references therein). Applications of the HB in finite dimensional dynamical systems often arise in the study of population-based models, for instance in predator-prey systems featured by the Allee effect [3]; in chemical systems, as in the Schnakenberg's [4]; in the study of coupled systems near a supercritical HB [5]; in the theory of electronic circuits [6]; in mathematical economics [7]; in the modeling of mechanical systems [8]; even in the study of travelling waves phenomena [9] and, generally speaking, almost whenever the considered model shows non-linear oscillations. Remarkable applications of HB in fluid dynamics can be found in [1] and references therein. In addition, the topic is extensively treated in the theory of bifurcations with higher codimension, as in the Turing-Hopf instabilities [10]. Nevertheless, in order to focalize the scenario, we suggest in this paper to split the classification of degeneracy into two different kinds. Degeneracy of first kind shall mean that no limit cycle surrounding the steady state emerges neither after nor before the critical value of the bifurcation parameter, with the possible emergence of limit cycles surrounding the point at infinity [11]. The second kind of degeneracy shall mean that multiple limit

cycles or semistable cycles as a limiting case, emerge either at super- or at subcritical bifurcation surrounding the steady state. We focus our attention in such a system that, in a neighborhood of the origin, can be written in the form

$$\dot{X} = F(X, a) \quad (1)$$

where

$$F = \sum_{k+l=1}^R \begin{pmatrix} \sigma_{kl}^1 \\ \sigma_{kl}^2 \end{pmatrix} x^k y^l + O(\|X\|^{R+1}) \quad (2)$$

is the Taylor expansion of  $F$  in a neighborhood of the origin, the coefficients  $\sigma_{kl}^m = \sigma_{kl}^m(a)$  are real, and  $X = (x, y)^T$ . Here  $a$  is a real parameter, and  $F$  is at least a  $C^0$ -function of  $a$ . Any smooth dynamical system in a vicinity of an isolated singular point  $P_a$  on the plane, can be rewritten in the form Eq.1 by a parameter-dependent shift of coordinates which relocates the singular point  $P_a$  of the system into the origin of coordinates. Eq.1 is called the system *in variations* corresponding to a given former system near the singular point  $P_a$ . Any reference to the steady state  $P_a$  of the former system will be associated to the punctual orbit at the origin of the Eq.1. The vector field with polynomial components of degree  $R$  represented in the main part in Eq.2, is called the *R-jet* of Eq.1 around the origin, which is usually denoted

$j_R(F)(0)$ . Two smooth vector fields defined in a neighborhood of the origin are called *R-jet equivalent* if their *R*-jet coincide.

Dynamical systems featured by an HB often arise in mathematical models for biological or chemical reaction systems [4]. In general, these models lead to polynomial vector fields having fifth degree at most. Moreover, some complex interactions between the different components in the system may lead to a representation in which a rational fraction with linear or quadratic denominator appear in the former system. For instance, this is the case in the Michaelis-Menten kinetics or in the prey-predator Holling systems in presence of Allee effect [3]. In such situations, after rescaling the temporal variable, the system can be transformed into an orbitally equivalent [12] polynomial system of degree at most six.

We recall that only few combinations of coefficients corresponding to odd terms in the reaction system can contribute to the emergence of limit cycles at bifurcation. We have called these the *Hopf coefficients*. These Hopf coefficients can be easily calculated and will play a similar role than such the Lyapunov coefficients do [13]. We suggest a new classification of degenerate bifurcations in terms of the Hopf coefficients, through the “discriminant” introduced in [10]. In this direction, we introduce the concepts of degenerate

HB of first and second kind.

In accordance with the proposed classification we subdivide the HB Theorem, in order to emphasize differences in the resulting behavior at bifurcation. We prove that, at degenerate bifurcation of first kind no limit cycle surrounding the steady state emerges, being allowed the emergence of limit cycles surrounding the point at infinity. At supercritical (subcritical) degenerate HB of second kind multiple or semistable limit cycles emerge surrounding the steady state. In correspondence to the procedure in [10], we shall see that, at a non-degenerate HB, a single limit cycle emerges. Each kind of degenerate HB is complemented with appropriate examples.

The period of the emerging limit cycles in one-parameter bifurcations attract the attention of researchers (see [11]) showing that, the main term in the asymptotic of the period of the emerging periodic solution, characterizes the bifurcation. Here we focus our attention in a procedure that gives us simultaneously an asymptotic estimate of the radius of the cycle and the frequency of the corresponding periodic orbit. As we shall see, the radius of the limit cycles and the period of the periodic solutions emerging at degenerate HB of second kind, tends to zero with an “anomalous” order as the bifurcation parameter tends to the critical value.

The plan of the paper is as follows. In Section 2 we summarize previous results in the treatment of an HB taken basically from [10]. In Section 3, the notion of discriminant is quoted, which is used in Definition 3 to classify HB taking into account its asymptotic behavior as the trace of the Jacobian tends to zero. The splitted version of the HB Theorem in 1, 2 and 3 in this Section describe the emergence of limit cycles in non-degenerate, degenerate of first and of second kind respectively. Finally, in Section 4 are gathered "typical" forms corresponding to each type of degenerate bifurcation. We include a sufficient condition for degenerate HB of second kind of Eq.1 valid up to 6-jet-equivalence .

## 2 Preliminaries

Let  $\delta_a$  and  $\tau_a$  be the determinant and the trace

$$\delta_a = \det(J_a) \tag{3}$$

$$\tau_a = \text{trace}(J_a) \tag{4}$$

of the Jacobian matrix

$$J_a = \begin{pmatrix} \sigma_{10}^1 & \sigma_{01}^1 \\ \sigma_{10}^2 & \sigma_{01}^2 \end{pmatrix} \quad (5)$$

at the origin  $O$  of the function  $F$  in Eq.1. The subindex in Eqs.3 and 4 indicates a functional dependence respect to  $a$ , which varies in an open small neighborhood  $U$  of the point  $a_*$  at which the trace vanish to change its sign. We shall assume that, the function  $a \rightsquigarrow \tau_a$  is an homeomorphism between  $U$  and a neighborhood  $V$  of  $\tau_a = 0$ , so the transversality condition  $\tau'_a(a_*) \neq 0$  is not required. Consequently, the parameter  $\tau_a$ ,  $|\tau_a| \ll 1$ , can be considered as the *intrinsic* bifurcation parameter (see [10]) and,  $\tau_a = 0$  is the critical value. The HB appears provided the inequality

$$\tau_a^2 - 4\delta_a < 0 \quad (6)$$

holds for any value  $\tau_a$  in some neighborhood of  $\tau_a = 0$ . Then, for every value  $a \in U$ , or equivalently, for every value  $\tau_a \in V$ , we have

$$\delta_a > 0 . \quad (7)$$



Further, if  $\tau_a < 0$  (respect.  $\tau_a > 0$ ) the origin is a stable (respect. unstable) focus. The bifurcation is *subcritical* if there is a limit cycle emerging for negative values of  $\tau_a$  close enough to zero. The bifurcation is *supercritical* if there is a limit cycle emerging for positive small values of  $\tau_a$ . From Eq.6 follows  $\sigma_{01}^1 \cdot \sigma_{10}^2 < 0$  and we do not loose generality assuming  $\sigma_{01}^1 < 0$ . Besides, there is no added restriction if we consider the Jacobian matrix in the system Eq.1 to have the simplest form:

$$J_a = \frac{1}{2} \begin{pmatrix} \tau_a & -\Lambda_a \\ \Lambda_a & \tau_a \end{pmatrix} \quad (8)$$

where

$$\Lambda_a = +\sqrt{4\delta_a - \tau_a^2} > 0$$

in a neighborhood of  $\tau_a = 0$ . So, we shall assume in the following that the Jacobian in Eq.5 has the form Eq.8. If necessary, the system Eq.1 can be rewritten through a linear transformation of variables in order that the Jacobian has the required form Eq.8, whenever the condition Eq.6 holds.

## 2.1 Averaging Hopf periodic solutions

Let us quote in this Subsection some results about the procedure in the study of HB proposed in [10]. Following that paper, we rewrite the system Eq.1, in the form

$$\dot{X} = J_a X + \Psi(X) \quad (9)$$

where  $X = (x(t), y(t))^T$ , and the vector function  $\Psi$  is given by the Taylor expansion of the difference

$$\Psi(X) = F(X, a) - J_a X$$

so,  $\Psi(X)$  contains all nonlinearities. In the function  $\Psi(X)$  we include the Taylor terms until a required precision, say up to  $R$ -jet-equivalence, together with the corresponding remainder. First we assume that  $\Psi(X)$  is analytical and a bit later, in Remark 1, we turn back into the smooth case. In [10], the authors proposed an algorithm allowing the reduction of an analytical reaction system showing an HB into a second order differential equation representing a weakly nonlinear oscillator in normal form. This transform of variables is analytical and nonlinear in general, but it was proved there that it is enough to consider the linear part of this transform to obtain the

equation of the oscillator preserving the required accuracy. The main idea of this procedure is that the transform of variables can be taken “close” to the appropriate linear transform in a neighborhood of the origin. So we quote the procedure in [10], and consider an invertible analytical transform of variables between neighborhoods of the origin

$$Y = \mathcal{H}(X) = \Gamma X + \mathcal{G}(X) \quad (10)$$

where  $\Gamma = (\gamma_{ij})$  is a non-singular matrix, and  $\mathcal{G}(X)$  be analytical without linear terms. By the Inverse Function Theorem, the existence of the inverse is guaranteed because  $\Gamma$  is non-singular and  $\mathcal{H}$  has smooth continuous derivatives. The inverse to Eq.10 has the form

$$X = \mathcal{H}^{-1}(Y) = \Gamma^{-1}Y + \mathcal{K}(Y) . \quad (11)$$

**Definition 1** *We say that the diffeomorphisms  $\mathcal{H}$  and  $\Gamma$  have a contact at the origin of order  $S \in \mathbb{N}$ ,  $S \geq 1$ , if*

$$\mathcal{H}(X) - \Gamma X = O\left(\|X\|^{S+1}\right) \quad (12)$$

as  $\|X\| \rightarrow 0$ .

We would like to substitute the function  $\mathcal{H}$  in Eq.10 by an equivalent simpler one, say  $\Gamma$ , in the procedure of rewriting the system Eq.1 in new easy-handled variables to be used in Proposition 1. In this endeavor we get more precision as the order of contact  $S$  between  $\mathcal{H}$  and  $\Gamma$  be greater. Further, from Eq.12 follows that  $\mathcal{H}^{-1}$  exists whenever  $\Gamma$  is invertible, and it can be also concluded that  $\mathcal{H}^{-1}$  and  $\Gamma^{-1}$  have a contact of order  $S$ . Following the ideas in [10] it can be proved that, if  $\Gamma$  satisfies certain “concordance” condition and  $\mathcal{H}$  has a contact with  $\Gamma$  at the origin of order  $R$ , then  $Y = \mathcal{H}(X)$  represents an analytical transform of coordinates such that every solution  $(x(t), y(t))^T$  to the analytical system Eq.1 is transformed into the form

$$Y = \left( z(t), \dot{z}(t) \right)^T \quad (13)$$

being  $z(t)$  unknown. Hence, the integration of the system Eq.9 can be reduced to the integration of a second order differential equation in the variable  $z$ . Consequently, rather than the exact expressions of the functions  $\mathcal{G}(X)$  and  $\mathcal{K}(Y)$ , it is sufficient to take the linear transforms  $\Gamma$  and  $\Gamma^{-1}$  instead of  $\mathcal{H}$  and  $\mathcal{H}^{-1}$  in the derivation of the equation in  $z$ , considering that this

substitution preserves the required accuracy. More precisely, for a given  $R$  in Eq.2 we may take a transform of coordinates  $\mathcal{H}$  which has a contact of order  $R$  with  $\Gamma$  at the origin. In the following statement we quote a result in [10, Prop.1] in which are gathered all the above ideas:

**Proposition 1** *Let us assume that Eq.6 holds. Then, there exists an invertible analytical transform of variables Eq.10 in the system Eq.9 such that Eq.13 holds. The matrix  $\Gamma$  is any non-trivial linear combination of the pair*

$$\Gamma_1 = \begin{pmatrix} 1 & 0 \\ \sigma_{10}^1 & \sigma_{01}^1 \end{pmatrix} ; \Gamma_2 = \begin{pmatrix} 0 & 1 \\ \sigma_{10}^2 & \sigma_{01}^2 \end{pmatrix} . \quad (14)$$

*The function  $z$  in Eq.13 satisfies the following second order equation:*

$$\ddot{z} - \tau_a \dot{z} + \delta_a z = G(z, \dot{z}) \quad (15)$$

*where the right hand side in Eq.15 does not involve linear terms in  $z, \dot{z}$ . More precisely,*

$$\begin{aligned} G(z, \dot{z}) = & \Pi_2 \{ \Gamma (J_a \mathcal{K}(Y) + \Psi(\mathcal{H}^{-1}Y)) \\ & + \langle \text{grad}_X \mathcal{G}, \mathcal{F} \rangle (\mathcal{H}^{-1}Y) \} \end{aligned} \quad (16)$$

being  $\Pi_2$  the standard projector over the second component.

So, the function in Eq.16 can be expanded asymptotically by

$$G\left(z, \dot{z}\right) = \Pi_2 \left\{ \Gamma \left( \Psi \left( \Gamma^{-1} Y \right) \right) \right\} + O\left(\|Y\|^{R+1}\right) . \quad (17)$$

Let us now return to Eq.15. We shall look for an oscillation with positive and small, but finite, amplitude  $\varepsilon$ . The small parameter  $\varepsilon$  is connected with the small bifurcation parameter  $\tau_a$  and will be defined a bit later. Taking in Eq.15 the change of variables

$$z(t) = \varepsilon \varsigma(t) \quad (18)$$

we get the equation of a weakly nonlinear oscillator in *normal form*:

$$\ddot{\varsigma} - \tau_a \dot{\varsigma} + \delta_a \varsigma = \varepsilon G\left(\varsigma, \dot{\varsigma}; \varepsilon\right) . \quad (19)$$

Then, to each periodic solution to Eq.9 will correspond a non-trivial periodic solution to Eq.19. In [10], was considered the Krylov-Bogoliubov averaging method [14] to derive an asymptotic expansion to the solution of Eq.19. To do so, let us consider the new variables  $r = r(t)$  and  $\theta = \theta(t)$  defined as

follows

$$\varsigma = r \cos (t + \theta) \quad (20)$$

$$\dot{\varsigma} = -r \sin (t + \theta) \quad (21)$$

then, the corresponding averaged equations are

$$\dot{r} = -\frac{1}{2\pi} \int_0^{2\pi} \sin \phi \{ -\tau_a r \sin \phi + \varepsilon G(r \cos \phi, -r \sin \phi; \varepsilon) \} d\phi \quad (22)$$

$$\dot{\theta} = -\frac{1}{2\pi r} \int_0^{2\pi} \cos \phi \{ -\tau_a r \sin \phi + \varepsilon G(r \cos \phi, -r \sin \phi; \varepsilon) \} d\phi \quad (23)$$

thus,

$$\dot{r} = \frac{r}{2} \{ \tau_a - p(r; \varepsilon) \} \quad (24)$$

$$\dot{\theta} = q(r; \varepsilon) \quad (25)$$

where

$$p(r; \varepsilon) = \frac{\varepsilon}{\pi r} \int_0^{2\pi} \sin \phi G(r \cos \phi, -r \sin \phi; \varepsilon) d\phi \quad (26)$$

$$q(r; \varepsilon) = -\frac{\varepsilon}{2\pi r} \int_0^{2\pi} \cos \phi G(r \cos \phi, -r \sin \phi; \varepsilon) d\phi . \quad (27)$$

Let us quote now some important properties about the functions  $p$  and  $q$  above.

**Proposition 2** *Functions  $p(r; \varepsilon)$  and  $q(r; \varepsilon)$  have at least order  $O(\varepsilon^2)$  or, equivalently,  $p(r; \varepsilon)/r^2$  and  $q(r; \varepsilon)/r^2$  have a finite limit as  $r \rightarrow 0$ . Moreover, the Taylor expansions of  $p(r; \varepsilon)$  and  $q(r; \varepsilon)$  must not contain odd powers of  $r$ .*

From Proposition 2 it can be concluded that the development of  $p$  has the form

$$p(r; \varepsilon) = p_3 \varepsilon^2 r^2 + p_5 \varepsilon^4 r^4 + \dots \quad (28)$$

in which  $p_s = p_s(\tau_a)$ , as a consequence of the dependence that the right hand side of Eq.1 has on the bifurcation parameter  $\tau_a$ . In Subsections 4.1, 4.2 and 4.3 can be found some examples showing this type of dependence.

We recall now the fact that, in accordance with Eqs.24 and 28, the  $M$  coefficients  $(p_{2j+1})_{j=1}^M$  will determine the bifurcation up to  $2(M+1)$ -jet-equivalence. So, it can be expected that we would manage the coefficients  $p_{2j+1}$  by taking  $M$  appropriate independent relations involving the parameters in the system Eq.1. This scenario is called a codimension- $M$  bifurcation. In accordance with our purpose in this paper, let us now introduce a defini-



tion that will be essential for the classification of HB. The reason to take it, will arise in the next Section.

**Remark 1** *We also recall that we are considering smooth vector fields, while in the construction of the analytical transform of coordinates in Eq.10 we use analytical properties. But notice that the transform Eq.10 as well as the appropriate  $R$ -jet approximation, which is analytical, lead to functions Eq.17 in which the principal part remains unaltered after their substitutions into the smooth system Eq.1.*

**Definition 2** *Let  $p_{2s+1}$  be a coefficient in the formal development Eq.28, which is derived from the formal  $\infty$ -jet of  $F$ . It shall be called negligible if satisfies*

$$|p_{2s+1}| \leq K_s |\tau_a| \quad (29)$$

*for certain constant  $K_s > 0$  as  $\tau_a \rightarrow 0$ . The function  $p(r; \varepsilon)$  in Eq.28 is said to be negligible if for all  $s \in \mathbb{N}$  the coefficient  $p_{2s+1}$  is negligible.*

Naturally,  $p_{2s+1} \equiv 0$  and  $p \equiv 0$  are respectively included in the above definition. For instance,  $p \equiv 0$  if in the formal  $j_\infty(F)(0)$  the non-vanishing terms have even degree. Moreover, we can get  $p_{2s+1} \equiv 0$  in spite of the existence of non-zero coefficients with degree  $2s + 1$  in the formal Taylor

development of the right hand in Eq.1. As we shall see in the next Section, negligible terms have no influence in the generation of limit cycles. Besides, with this notion we are able to give a more detailed version of the Proposition 3 in [10] as follows.

**Proposition 3** *If the function  $p(r; \varepsilon)$  is non-negligible, there must exist a positive integer  $N$  and a positive real value  $r_0 = r_0(\tau_a)$  such that  $p(r, \varepsilon)$  has the non-trivial Taylor expansion:*

$$p(r; \varepsilon) = \omega \varepsilon^{2N} r_0^{-2N} r^{2N} + O(\varepsilon^{2N+2} r^{2N+2}) \quad (30)$$

where  $\omega = +1$  or  $-1$ . In addition, the behavior of the factor  $r_0^{-2N}$  as  $\tau_a \rightarrow 0$  obeys the following alternative: either

$$\lim_{\tau_a \rightarrow 0} r_0^{-2N} = r_*^{-2N} > 0 \quad (31)$$

or, for a given  $\gamma$ ,  $0 < \gamma < 1$ ,

$$r_0^{-2N} = O_S(|\tau_a|^\gamma) \text{ as } \tau_a \rightarrow 0. \quad (32)$$

As in [15], the symbol  $O_S$  in Eq.32 means a sharp estimate, that is:  $r_0^{-2N} = O(|\tau_a|^\gamma)$  and  $r_0^{-2N} \neq o(|\tau_a|^\gamma)$  as  $\tau_a \rightarrow 0$ . We recall that the bifurcation is supercritical (respect. subcritical) if  $\omega = +1$  (respect.  $\omega = -1$ ). In the supercritical case, the root  $r_0$  appears for  $\tau_a > 0$  (respect.  $\tau_a < 0$ ), so the limit in Eq.31 or the order relation in Eq. 32 should be considered as  $\tau_a \rightarrow 0+$  (respect.  $\tau_a \rightarrow 0-$ ). Consequently,

**Proposition 4** *Let us assume that  $p(r; \varepsilon)$  be non-negligible and also, that  $r_0$  in Eq.30 has the property in Eq.31. Hence, there is a positive root  $\rho$  to the equation*

$$p(r; \varepsilon) - \tau_a = 0 \quad (33)$$

*either for positive or for negative values  $\tau_a$  close enough to zero. Furthermore, up to the leading term, the root to Eq.33 has the form*

$$\rho = \left( \frac{|\tau_a|}{\varepsilon^{2N}} \right)^{1/2N} (r_* + O(|\tau_a|)) + O(\varepsilon^2) . \quad (34)$$

**Proof.** The property Eq.31 is equivalent to  $r_0^{-2N} = r_*^{-2N} + O(|\tau_a|)$  and, also to  $r_0 = r_* + O(|\tau_a|)$ . ■

Due to Proposition 4 the function  $p$  in Eq.26 has been called a *discriminant* for the HB in [10]. Let us assume the existence of a (finite) positive

root Eq.34 such that Eq.31 holds. Then, the small parameter  $\varepsilon$  introduced in Eq.18 can be taken as

$$\varepsilon^{2N} = |\tau_a| \quad . \quad (35)$$

From Eq.35 and the relation  $|\tau_a| = O\left(|\tau_a|^{1/N}\right)$  as  $\tau_a \rightarrow 0$  for  $N \geq 1$ , it follows that Eq.34 can now be written as

$$\rho = r_* + O\left(|\tau_a|^{1/N}\right) \quad . \quad (36)$$

If Eq.31 does not hold we may assume Eq.32, so in a similar way as we do in Proposition 4 to get Eq.35, we now arrive to

$$\varepsilon^{2N} = |\tau_a|^{1-\gamma} \quad . \quad (37)$$

Moreover, if for instance

$$r_0^{-2N} = r_L |\tau_a|^\gamma + o(|\tau_a|^\gamma) \quad \text{as } \tau_a \rightarrow 0$$

for certain positive number  $r_L$ , then Eq.34 can be rewritten as

$$\rho = |\tau_a|^{(1-\gamma)/2N} \frac{r_L}{\varepsilon} + O(\varepsilon^2)$$

and follows

$$\rho = r_L + O\left(|\tau_a|^{(1-\gamma)/N}\right) . \quad (38)$$

The hypothesis of the HB Theorem, as appear in [10, Theorem 1], contains an implicit reference to the non-degenerate case, more precisely, to the condition Eq.31. As we shall see in Section 3, Eqs.31, 32 and 29 put in evidence the reason for differentiation in HB we are suggesting here. In accordance with the intention and terminology in this paper, we shall give in Section 3 a split version of the HB theorem.

### 3 Theorems for degenerate Hopf bifurcation

Basically, the standard classification of HB is conformed by two main classes: while the system at degenerate HB shows a center at the critical value  $a_*$ , the system at a non-degenerate HB shows a weak focus at the critical value, leading to the emergence (super- or subcritical) of a limit cycle [4]. When the bifurcation occurs in a one-parameter family of vector fields whose first non-zero derivatives at the origin have order  $2N + 1$ ,  $N > 1$ , it is called [11] a generalization of the Andronov-Hopf's. Other higher codimension HB,

for instance the Bautin bifurcation, are often called generalized [12]. In our formulation, we shall include such higher codimension HB, hence we shall assume we have a family of systems parametrized by the bifurcation parameter  $\tau_a$ . Consequently, the coefficients in Eq.28 also depend on the bifurcation parameter,  $p_s = p_s(\tau_a)$ , so we can *a priori* classify the HB taking into account how these dependences are.

**Definition 3** *We shall say that the Hopf bifurcation is degenerate of first kind if  $p$  is negligible (See Definition 2 ). Let  $N$  be given as in Eq.30. The bifurcation shall be called degenerate of second kind, if there exists a number  $\gamma$ ,  $0 < \gamma < 1$ , such that Eq.32 holds. The HB shall be called non-degenerate, provided Eq.31.*

For instance, if  $p \equiv 0$  in Eq.28, the bifurcation shall be degenerate of first kind. Further, the existence of at least one non-negligible  $p_{2s+1}$  derived from the formal  $\infty$ -jet of  $F$ , no matter how large the number  $s$  is, implies that the HB will not be degenerate of first kind. We also remark that, at degenerate bifurcation, we are implicitly considering that the system moves close to a higher codimension point as the bifurcation parameter varies, because the non-zero coefficients  $p_{2s+1}$  vanish at  $\tau_a = 0$ .

**Remark 2** *There is a reason to take  $0 < \gamma < 1$  in the definition of the degeneracy of second kind above. As  $F$  in Eq.1 is a continuous function of  $\tau_a$  in a vicinity of  $\tau_a = 0$ , we may assume  $\gamma > 0$  as a consequence of Eq.32. Furthermore, from Eq.37 and the “smallness” of the parameter  $\varepsilon$  in Eq.18 as  $\tau_a \rightarrow 0$  follows that  $1 - \gamma > 0$ .*

It is easy to see, from Definition 3, that degenerate (of any kind) HB implicitly implies that the system shows a center at the critical value of the bifurcation parameter. This is because the main terms of  $p$  in Eq.24 tends to zero as  $\tau_a \rightarrow 0$ , so the origin is a center at the critical value. We split up the HB theorem taking into account each type of bifurcation. We recall that the bifurcation is supercritical if the cycle emerges provided  $0 < \tau_a \ll 1$ , and subcritical if it emerges for  $0 < -\tau_a \ll 1$ .

**Theorem 1 (non-degenerate Hopf bifurcation)** *Let us assume that Eq.6 holds and that Eq.33 has a root Eq.36 with the property Eq.31 for positive (respectively, negative) but sufficiently close to zero values of the bifurcation parameter  $\tau_a$ . Then, a single limit cycle to the system in Eq.9 emerges. Furthermore, the limit cycle is orbitally asymptotically stable (respect., unstable) if and only if the bifurcation is supercritical (respect., subcritical).*

The radius of the emerging cycle is  $r = O_S \left( |\tau_a|^{1/2N} \right)$ , while the frequency is  $\varpi = 1 + O \left( |\tau_a|^{1/N} \right)$  as  $\tau_a \rightarrow 0$ .

**Proof.** The proof follows from Eq.33 and Eq.24. Let the main term in Eq.30 have the property Eq.31. Then, a single root to Eq.33 tends to zero as  $\tau_a \rightarrow 0$ . This fact means that a single limit cycle emerges at bifurcation. It is not excluded the existence of further different roots to Eq.33, but if any other appears, it determines a limit cycle that does not vanish in a vicinity of  $\tau_a = 0$ , so the cycle “persists” along the bifurcation. As follows from Eqs.54, 55 and 56, we get the order of the radius of the limit cycle. The order of the frequency is determined in Subsection 3.3. ■

Consequently, a necessary but not sufficient condition for the emergence of a limit cycle at bifurcation, is the existence of nonzero odd order terms in the expansion of the components in the right-hand side of Eq.1. For instance, the following result states that the condition is not sufficient.

**Theorem 2 (degenerate HB of first kind)** *Let us assume that Eq.6 holds and assume  $p$  negligible (Definition 2). Then, neither limit cycle surrounding the steady state emerges after nor before the critical value.*



**Proof.** From Eq.29 and Eq.33 follows that none of the roots to Eq.33 tends to zero as  $\tau_a \rightarrow 0$ . More precisely, if any root exists, it tends to infinity. Taking into account only the leading terms we may write

$$p_{2k+1} = \alpha_k \tau_a^{1+\mu_k}$$

where  $\mu_k \geq 0$ ,  $\alpha_k$  are constants which can be zero. Let  $N$  corresponds to the first non-zero  $\alpha_k$ , and take  $\varepsilon^{2N} = \tau_a$  (if the bifurcation is supercritical), so Eq.28 is

$$p = \varepsilon^{2N} \sum_{n=N}^{\infty} \alpha_k \tau_a^{1+\mu_k} \varepsilon^{2(n-N)} r^{2N} = \tau_a^2 \sum_{n=N}^{\infty} \alpha_k \tau_a^{\mu_k+(n-N)/N} r^{2N}$$

hence, if there is a root to  $\tau_a - p(r; \varepsilon) = 0$  then it should tend to infinity as  $\tau_a \rightarrow 0$ . ■

**Remark 3** *We remark that, at a degenerate HB of first kind, further limit cycles may persist in a neighborhood of the critical value of the bifurcation parameter as can be seen in Subsection 4.3. Furthermore, if the order of a negligible term  $p_{2s+1}$  is greater than one, then limit cycles surrounding the point at infinity may emerge in the so-called HB at the infinity (see [11] and references therein). In Subsection 4.3 we shall show an example of this*

situation. For instance, this is the case if

$$p_{2s+1} = A \tau_a^{\beta+1} + o\left(|\tau_a|^{\beta+1}\right)$$

being  $\tau_a \rightarrow 0$ ,  $\beta > 0$ ,  $A \neq 0$ .

Now, with the same procedure in the proof of Theorem 1, we have,

**Theorem 3 (degenerate HB of second kind)** *Let us assume that Eq.6 holds and that Eq.33 has a root Eq.36 with the property Eq.32 for positive (respectively, negative) but sufficiently close to zero values of the parameter  $\tau_a$ . Then, it can be assured the emergence of at least one limit cycle to the system in Eq.1, the radius of which has order*

$$r = O_S\left(|\tau_a|^{(1-\gamma)/2N}\right) \quad (39)$$

while the frequency is  $\varpi = 1 + O\left(|\tau_a|^{(1-\gamma)/N}\right)$  as  $\tau_a \rightarrow 0$ . The number  $\gamma$ ,  $0 < \gamma < 1$  corresponds to the one in Eq.32.

**Proof.** The proof for the non-degenerate bifurcation can be repeated, but taking the small parameter from Eq.37. Notice that in this case more than a single positive root to Eq.33 tending to zero as  $\tau_a \rightarrow 0$ , may appear. ■

At a degenerate HB of second kind, different behaviors can be observed: a single or several limit cycles may emerge, including semi-stable cycles as the limiting case at which different limit cycles collapse. In Section 4 some examples with this kind of degeneracy are considered to show that, multiple limit cycles or semistable limit cycles might appear either sub- or supercritically. Bearing in mind the construction of the examples in Section 4, looks easy to find sufficient conditions for the emergence of multiple limit cycles, as it is done in Subsection 4.5.

We can show sufficient conditions for the stability of the emerging limit cycles surrounding the steady state. Such conditions are based on the behavior of the discriminant  $p$  near the root to which the cycle corresponds.

**Proposition 5** *Consider a root  $\rho$  to Eq.33, so it corresponds to a limit cycle  $L$ . This cycle is asymptotically stable or unstable if the number  $dp/dr(\rho)$  is negative or positive respectively.*

**Proof.** In the calculation of  $dp/dr(\rho)$  we can assume  $\rho = r_0$  as in Eq.36, or  $\rho = r_L$  in Eq.38, in accordance with the type of bifurcation. Due to the continuity argument near  $\tau_a = 0$  in Eqs. 36 or 38, and the fact that  $dp/dr(r_0)$  (or  $dp/dr(r_L)$ ) does not vanish, the assertion follows. ■

### 3.1 The Hopf coefficients

Let us take  $M \geq 2$  in Eq.1 and the map Eq.10 in which  $\mathcal{H}$  has a contact with  $\Gamma$  of order  $M$  at the origin, we get

$$\dot{Y} = \sum_{1 \leq k+l \leq M} \Gamma \begin{pmatrix} \sigma_{kl}^1 \\ \sigma_{kl}^2 \end{pmatrix} \left( \mu_{11}z + \mu_{12}\dot{z} \right)^k \left( \mu_{21}z + \mu_{22}\dot{z} \right)^l + O\left(\left(\|Y\|^{M+1}\right)\right)$$

where  $\Gamma = (\gamma_{ij})$  and  $\Gamma^{-1} = (\mu_{ij})$ . For instance, if we take  $\Gamma = \Gamma_1$ , the quantities

$$\mu_{11} = 1 ; \mu_{12} = 0 ; \mu_{21} = \tau_a \Lambda_a^{-1} ; \mu_{22} = -2\Lambda_a^{-1} \quad (40)$$

are the components of  $\Gamma_1^{-1}$ . So, Eq.16 can be written

$$G\left(z, \dot{z}\right) = \sum_{2 \leq k+l \leq M} R_{kl} \left( \mu_{11}z + \mu_{12}\dot{z} \right)^k \left( \mu_{21}z + \mu_{22}\dot{z} \right)^l + O\left(\left(\|Y\|^{M+1}\right)\right) \quad (41)$$

where

$$R_{kl} = \gamma_{21} \sigma_{kl}^1 + \gamma_{22} \sigma_{kl}^2 . \quad (42)$$

Hence, Eq.41 can be rewritten in terms of powers of the  $Y$ -components

as

$$G(z, \dot{z}) = \sum_{2 \leq k+l \leq M} H_{kl} z^k (\dot{z})^l + O\left(\left(\|Y\|^{M+1}\right)\right). \quad (43)$$

Only the non-zero  $H_{mn}$  in Eq.43 corresponding to such pairs  $(m, n)$  for which  $K_{mn} \neq 0$ , where

$$K_{mn} = \int_0^{2\pi} \cos^m \phi \sin^{n+1} \phi d\phi \quad (44)$$

can contribute to the appearance of a nonzero term in Eq.30 so, to the appearance of a limit cycle solution.

Let us introduce the following

**Definition 4** *We shall call the Hopf coefficient of degree  $(2N + 1)$  to the coefficient  $p_{2N+1}$  in the expansion Eq.28, which is an algebraic combination of coefficients  $H_{mn}$  in Eq.43 provided  $m + n = 2N + 1$ .*

From Eq.21, Eq.43 and Eq.26 follows directly that

$$p_3 = -\frac{1}{4} (3H_{03} + H_{21}) \quad (45)$$

is the Hopf coefficient of third degree, and

$$p_5 = -\frac{1}{8} (5H_{05} + H_{23} + H_{41}) \quad (46)$$

is the Hopf coefficient of degree five. Other Hopf coefficients can be derived by forward calculations. We recall that are required only the coefficients of  $\Gamma$  and the coefficients of the main part of the Taylor expansion, to calculate the Hopf coefficients. Becomes easy to check from Eq.44 (see Appendix 6) that the coefficients in Eq.43 leading to Hopf coefficients in Eq.17 up to 6-jet equivalence are

$$H_{0,3} \quad H_{2,1} \tag{47}$$

$$H_{0,5} \quad H_{2,3} \quad H_{4,1} \quad .$$

In accordance with Eq.42, if the Jacobian matrix have the form Eq.8 and being  $\Gamma = \Gamma_1$ , we have the numbers

$$R_{mn} = \frac{1}{2} (\tau_a \sigma_{m,n}^1 - \Lambda_a \sigma_{m,n}^2) \tag{48}$$

We recall that the matrix  $\Gamma$ , as a linear combination of the matrixes in Eq.14, depends on the elements of the matrix  $J_a$  which is supposed to have the form in Eq.8. Then, taking  $\Gamma = \Gamma_1$  and  $\Gamma^{-1} = (\mu_{ij})$ , we get the following equalities

$$H_{03} \underset{\text{def}}{=} - (8 R_{03}) \Lambda_a^{-3} , \tag{49}$$

$$H_{21} \underset{\text{def}}{=} -2 \left( R_{21} \Lambda_a^2 + 2R_{12} \tau_a \Lambda_a + 3R_{03} \tau_a^2 \right) \Lambda_a^{-3} , \quad (50)$$

$$H_{05} \underset{\text{def}}{=} - (32 R_{05}) \left( \Lambda_a^{-5} \right) , \quad (51)$$

$$H_{23} \underset{\text{def}}{=} -8 \left( R_{23} \Lambda_a^2 + 4R_{14} \tau_a \Lambda_a + 10R_{05} \tau_a^2 \Lambda_a^2 \right) \Lambda_a^{-5} , \quad (52)$$

$$\begin{aligned} H_{41} \underset{\text{def}}{=} & -2 \left( R_{41} \Lambda_a^4 + 2R_{32} \tau_a \Lambda_a^3 + 3R_{23} \tau_a^2 \Lambda_a^2 \right. \\ & \left. + 4R_{14} \tau_a^3 \Lambda_a + 5R_{05} \tau_a^4 \right) \Lambda_a^{-5} , \end{aligned} \quad (53)$$

from Eq.43.

It is concluded in this Subsection that, up to 6-jet-equivalence, we can easily check the type of degeneracy the degenerate HB shows. Particularly, Eqs.45 and 46 give the full information for polynomial vector fields of degree at most six. In Section 4 we show different examples of degenerate HB within the class of polynomial vector fields of degree at most six.

### 3.2 Asymptotic expansions of the limit cycles

Going back to the substitutions given in Eq.18 and Eq.10, we could derive uniform asymptotic expansion to the solution of Eq.1. So, if the HB is non-degenerate, it is possible to develop the periodic solution  $\Theta(t) = (\bar{x}(t), \bar{y}(t))$

to Eq.1 generating the limit cycle, by

$$\bar{x}(t) = u_1(t) (|\tau_a|)^{\frac{1}{2N}} + O\left(|\tau_a|^{\frac{1}{N}}\right) \quad (54)$$

$$\bar{y}(t) = v_1(t) (|\tau_a|)^{\frac{1}{2N}} + O\left(|\tau_a|^{\frac{1}{N}}\right) \quad (55)$$

and

$$\begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix} = r_* \Gamma^{-1} \begin{pmatrix} \cos(\varpi t) \\ -\sin(\varpi t) \end{pmatrix} \quad (56)$$

with frequency

$$\varpi = 1 + q\left(\rho, |\tau_a|^{\frac{1}{2N}}\right) \quad (57)$$

and period  $T = \frac{2\pi}{\varpi}$ . From Eq.25 the angular speed of the oscillation is obtained. Note that, up to the leading terms, the expansions of the cycle solution in Eqs.54 and 55 are uniform, as the  $O\left(|\tau_a|^{\frac{1}{2N}}\right)$ -terms are bounded functions. The expansions in Eqs.54 and 55, can be taken also for the limit cycle that emerges at degenerate HB of second kind considered in Theorem 3, but substituting the exponent  $\frac{1}{2N}$  by  $\frac{1-\gamma}{2N}$ , and  $r_*$  by the  $r_L$  given in Eq.38.



### 3.3 On the period of the limit cycles

Here we give a brief comment about the period of the limit cycles, considering the interest in the topic [11]. The period of the emerging limit cycles in non-degenerate or degenerate of second kind HB can be determined from the formula for the frequency, given in Eq.57. To do so, it is necessary to consider Eqs.23, 27 and 43. Up to 6-jet-equivalence, the expansion results

$$q = -\frac{1}{8}\varepsilon^2\rho^2 \left( [3H_{30} + H_{12}] + \frac{1}{2}\varepsilon^2\rho^2 [5H_{50} + H_{32} + H_{14}] \right) + O(\varepsilon^6) .$$

The coefficients  $H_{kl}$  in the above formula, can be calculated in a similar way as the others in Eqs.49 to 53. If the HB is non-degenerate we have the relation Eq.35, while we have to consider Eq.37 if the bifurcation is degenerate of second kind. Further, the factor  $\rho$  take the values Eq.36 or Eq.38 in accordance with the type of bifurcation. The procedure yields to the following estimates of the frequency as  $\tau_a \rightarrow 0$ :

$$q = O\left(|\tau_a|^{\frac{1}{N}}\right) \tag{58}$$

for the emerging cycle in the non-degenerate case, and

$$q = O\left(|\tau_a|^{\frac{1-\gamma}{N}}\right) \quad (59)$$

if the bifurcation is degenerate of second kind. We remark the fact that, in the last two formulas, the order it is not necessarily sharp. In the examples in the next Section we also have done a reference about the period.

## 4 Typical forms in degenerate HB

The main concern in this Section are degenerate HB of first and second kinds. In Subsections 4.1 to 4.4 we shall study “typical” forms by using polar coordinates in order to give a familiar, more geometrical, description of the nature of the bifurcation. Notice that the examples in the referenced Subsections are not normal forms, because in these cases no genericity conditions (see [12]) are involved. It is not difficult to show that, using the averaging method we can get the same conclusions. In Subsection 4.5 we state, as an example of what can be expected in presence of a degenerate HB of second kind, a sufficient condition using the Hopf coefficients up to 6-jet-equivalence.

## 4.1 Multiple cycles in supercritical degenerate Hopf bifurcation of second kind

In this paragraph we shall give an example of a system which shows a degenerate HB of second kind in accordance with Definition 3:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a^3 & -1 \\ 1 & a^3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - a^2 (x^2 + y^2) \begin{pmatrix} x \\ y \end{pmatrix} + \frac{3}{16} a (x^2 + y^2)^2 \begin{pmatrix} x \\ y \end{pmatrix}. \quad (60)$$

Neither limit cycle surrounds the focus while the trace  $\tau = 2a^3$  is negative. At the critical value  $a = 0$ , the origin is clearly a center to the system Eq.60. Further, when the trace becomes positive the steady state becomes unstable and two small limit cycles emerge with radius

$$r_1 = \frac{2\sqrt{3}}{3}a^{1/2} \quad \text{and} \quad r_2 = 2a^{1/2}$$

respectively. Both cycles emerge due to the bifurcation and have radius  $O_S(\tau^{1/6})$  as  $\tau \rightarrow 0$ . The first cycle is stable, while the second is unstable. To check the above assertions we only need to rewrite the system Eq.60 in

polar coordinates. We get the system:

$$\begin{cases} \dot{r} = ar \left( a^2 - ar^2 + \frac{3}{16}r^4 \right) \\ \dot{\theta} = 1 \end{cases}.$$

From this example it can be concluded that, at degenerate bifurcation of second kind, several limit cycles may emerge. Of course, with the same idea, it is possible to build polynomial dynamical systems with higher degree  $2N+1$  showing the emergence of  $N$  different limit cycles at a super- or subcritical degenerate HB of second kind. The frequency of the emerging cycles is

$$\varpi = 1 + O\left(|\tau_a|^{1/3}\right)$$

as  $N = 1$  and  $\gamma = 2/3$  in Eq.59.

## **4.2 Semistable cycles in supercritical degenerate Hopf bifurcation of second kind**

The following system shows a supercritical degenerate bifurcation of second kind:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a^3 & -1 \\ 1 & a^3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 2a^2 (x^2 + y^2) \begin{pmatrix} x \\ y \end{pmatrix} + a (x^2 + y^2)^2 \begin{pmatrix} x \\ y \end{pmatrix} . \quad (61)$$

The origin is an stable focus without any surrounding limit cycle while the trace  $\tau = 2a^3$  is negative. At the critical value  $a = 0$ , the origin is clearly a center to the system Eq.61. The system shows a degenerate bifurcation of second kind in accordance with Definition 3. Further, when the trace becomes positive the steady state becomes unstable and, a single small limit cycle emerges with radius

$$r = a^{1/2} .$$

This limit cycle is the  $\omega$ -limit set of any orbit inside the circle with the exception of the origin, but it is the  $\alpha$ -limit set of any orbit outside the circle. The corresponding system in polar coordinates is:

$$\begin{cases} \dot{r} = ar (r^2 - a)^2 \\ \dot{\theta} = 1 \end{cases} .$$

From this example it can be concluded that, at a degenerate bifurcation, semistable limit cycles may emerge. The radius of the limit cycle is  $r = O_S(\tau^{1/6})$  as  $\tau \rightarrow 0$ . The frequency of the emerging cycle is

$$\varpi = 1 + O(|\tau_a|^{1/3})$$

as  $N = 1$  and  $\gamma = 2/3$  in Eq.59.

### 4.3 Degenerate Hopf bifurcation of first kind without emergence of limit cycle

1. Let us first consider a system showing a degenerate HB of first kind:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - a(x^2 + y^2) \begin{pmatrix} x \\ y \end{pmatrix} \quad (62)$$

The origin is an stable focus without limit cycle surrounding while  $a < 0$ . At the critical value  $a = 0$  the origin is a center, and the steady state turns unstable if  $a > 0$ . It can be noted the existence of a limit cycle with radius  $r = 1$ , but this cycle is not a consequence of the bifurcation because it persists along the bifurcation. The polar system

is:

$$\begin{cases} \dot{r} = ar(1 - r^2) \\ \dot{\theta} = 1 \end{cases}.$$

showing that the cycle  $r = 1$  is a stable limit cycle for  $a > 0$ , changing its stability in dependence of the sign of  $a$ . This circle is still an orbit of Eq.62 for  $a = 0$ .

2. We recall in the fact that, the persisting limit cycle may be semistable, as in the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 2a(x^2 + y^2) \begin{pmatrix} x \\ y \end{pmatrix} + a(x^2 + y^2)^2 \begin{pmatrix} x \\ y \end{pmatrix} \quad (63)$$

which also shows a degenerate bifurcation of first kind. The corresponding polar system is:

$$\begin{cases} \dot{r} = ar(1 - r^2)^2 \\ \dot{\theta} = 1 \end{cases}$$

so, the limit cycle  $r = 1$  is semistable, its interior stability changes with the sign of  $a$ , and it is still an orbit of Eq.63 for  $a = 0$ .

## 4.4 Degenerate HB of first kind showing limit cycles at infinity

Let us consider now a system showing a degenerate HB of first kind, but leading in this case to the so called HB at infinity [11]:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - a a^\beta (x^2 + y^2) \begin{pmatrix} x \\ y \end{pmatrix} \quad (64)$$

$\beta > 0$ . Here we take  $a^\beta = \text{sign}(a) \cdot |a|^\beta$ . The portrait becomes clear taking polar coordinates:

$$\begin{cases} \dot{r} = ar (1 - a^\beta r^2) \\ \dot{\theta} = 1 \end{cases}.$$

We conclude that, as a consequence of the degenerate bifurcation of the system Eq.64 no limit cycle emerges surrounding the origin, but an unstable limit cycle surrounding the point at infinity, whose radius is  $r = a^{-\beta/2}$ , emerges supercritically. More precisely, the cycle emerges for small positive values of the trace  $\tau = 2a$ .



## 4.5 A sufficient condition for degenerate Hopf bifurcation of second kind

We shall consider here a degenerate HB for the system Eq.1 up to 6-jet-equivalence. The following statement, which is inspired in the examples in Subsections 4.1 and 4.2, gives a sufficient condition for the existence of degenerate bifurcations of second kind. The idea in this assertion is very simple.

**Proposition 6** *Let Eq.6 holds for the system Eq.1 at the origin, and let  $j_6(F)(0)$  be such that the coefficients  $p_3$  in Eq.45 and  $p_5$  in Eq.46 satisfy the conditions:*

$$p_3 = \tau_a^\gamma Q_3 + o(\tau_a^\gamma) ; p_5 = -\tau_a^{2\gamma-1} Q_5 + o(\tau_a^{2\gamma-1})$$

*for some  $\gamma$  ( $1/2 < \gamma < 1$ ) as  $\tau_a \rightarrow 0$ , where  $Q_3, Q_5$  are both positive numbers and*

$$\Delta = Q_3^2 - 4Q_5 \geq 0 .$$

*Then, two different limit cycles if  $\Delta > 0$ , or one semistable limit cycle if  $\Delta = 0$ , emerge at the supercritical degenerate bifurcation. The radius of the*

*cycles are  $O_S\left(\tau_a^{(1-\gamma)/2}\right)$  as  $\tau_a \rightarrow 0+$ .*

**Proof.** Taking  $1/2 < \gamma < 1$  in Eq.32 follows Eq.37, hence  $\varepsilon = \tau_a^{(1-\gamma)/2}$ . Up to the  $O(\tau_a)$  leading term in Eq.33, we get

$$Q_5 r^4 - Q_3 r^2 + 1 = 0$$

meaning that the algebraic equation above has either two different positive roots  $\rho_k$  ( $k = 1, 2$ ), or a single positive root with multiplicity two, depending on  $\Delta$ . Then, the proof follows from Eq.33 and Eq.24. ■

## 5 Conclusions

To study degenerate Hopf bifurcation in smooth dynamical systems near an isolated singular point on the plane we first focus in a classification of such bifurcations via a discriminant function. If the bifurcation is non-degenerate, then a single limit cycle emerges. The radius of this cycle can not be a priori supposed to have order  $O\left(\tau_a^{1/2}\right)$  as  $\tau_a \rightarrow 0$ , but in general have order  $O\left(\tau_a^{1/2N}\right)$ , for some integer  $N$ . In this scenario, the emerging limit cycle may coexist with other cycles which persist along the bifurcation. This classification gives some light to the question of whether degenerate bifurcations lead

to the emergence of limit cycles. We show that, no limit cycle surrounding the steady state emerges neither at super- nor at subcritical HB when it is degenerate of first kind. In this scenario, limit cycles surrounding the point at infinity may also emerge. For a degenerate HB of second kind, we found the anomalous asymptotic order of the radius and period of the emerging limit cycles, and further, we give a sufficient condition to the appearance either of a couple of limit cycles or one semistable cycle of the system Eq.1, up to 6-jet-equivalence. Finally, we propose some examples, which are called here “typical” forms, showing different behaviors that can occur at supercritical degenerate HB, being either of first or of the second kind.

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## 6 Appendix

In this Appendix we include the values of the integrals in Eq.44, which are used in the implementation of the Krylov-Bogoliubov averaging method. Further, we include the coefficients  $H_{mn}$  leading to Hopf coefficients. Notice that, for  $m + n \leq 6$  the non-zero numbers are:

$$K_{03} = \frac{3}{4}\pi ; \quad K_{21} = \frac{1}{4}\pi ;$$

$$K_{05} = \frac{5}{8}\pi ; \quad K_{23} = \frac{1}{8}\pi ; \quad K_{41} = \frac{1}{8}\pi .$$

For a given matrix  $\Gamma$ , which is a non-trivial linear combination of the pair in Eq.14, we denote  $\Gamma^{-1} = (\mu_{ij})$ . Then, the  $H_{mn}$  leading to Hopf coefficients up to 6-jet-equivalence are

$$\begin{aligned} H_{21} = & 3R_{30}\mu_{11}^2\mu_{12} + 2R_{21}\mu_{11}\mu_{12}\mu_{21} + R_{21}\mu_{11}^2\mu_{22} \\ & + 2R_{12}\mu_{11}\mu_{21}\mu_{22} + R_{12}\mu_{12}\mu_{21}^2 + 3R_{03}\mu_{21}^2\mu_{22} , \end{aligned}$$

$$H_{03} = R_{12}\mu_{12}\mu_{22}^2 + R_{03}\mu_{22}^3 + R_{30}\mu_{12}^3 + R_{21}\mu_{12}^2\mu_{22} ,$$

$$\begin{aligned}
H_{41} = & 2R_{32}\mu_{11}^3\mu_{21}\mu_{22} + 5R_{50}\mu_{11}^4\mu_{12} + R_{41}\mu_{11}^4\mu_{22} + 4R_{41}\mu_{11}^3\mu_{12}\mu_{21} \\
& + 2R_{23}\mu_{11}\mu_{12}\mu_{21}^3 + 3R_{23}\mu_{11}^2\mu_{21}^2\mu_{22} + 3R_{32}\mu_{11}^2\mu_{12}\mu_{21}^2 \\
& + R_{14}\mu_{12}\mu_{21}^4 + 4R_{14}\mu_{11}\mu_{21}^3\mu_{22} + 5R_{05}\mu_{21}^4\mu_{22} ,
\end{aligned}$$

$$\begin{aligned}
H_{23} = & 10R_{50}\mu_{11}^2\mu_{12}^3 + 4R_{41}\mu_{11}\mu_{12}^3\mu_{21} + 6R_{41}\mu_{11}^2\mu_{12}^2\mu_{22} \\
& + 6R_{32}\mu_{11}\mu_{12}^2\mu_{21}\mu_{22} + R_{32}\mu_{12}^3\mu_{21}^2 + 3R_{32}\mu_{11}^2\mu_{12}\mu_{21}^2 \\
& + 6R_{23}\mu_{11}\mu_{12}\mu_{21}\mu_{22}^2 + R_{23}\mu_{11}^2\mu_{22}^3 + 3R_{23}\mu_{12}^2\mu_{21}^2\mu_{22} \\
& + 6R_{14}\mu_{12}\mu_{21}^2\mu_{22}^2 + 4R_{14}\mu_{11}\mu_{21}\mu_{22}^3 + 10R_{05}\mu_{21}^2\mu_{22}^3 ,
\end{aligned}$$

$$\begin{aligned}
H_{05} = & R_{32}\mu_{12}^3\mu_{22}^2 + R_{23}\mu_{12}^2\mu_{22}^3 + R_{14}\mu_{12}\mu_{22}^4 \\
& + R_{05}\mu_{22}^5 + R_{50}\mu_{12}^5 + R_{41}\mu_{12}^4\mu_{22} ,
\end{aligned}$$

where  $R_{kl}$  are given in Eq.42.